ON THE CALCULATION OF FLOW PAST AXISYMMETRIC BODIES WITH DETACHED SHOCK WAVES USING AN ELECTRONIC COMPUTING MACHINE

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In solving the title problem the majority of authors have assumed the shape and position of the shock wave, after which the inverse problem was solved; a detailed review of existing methods is given in the paper [1]. The method of Dorodnitsyn [2] permits the direct problem to be solved with the necessary degree of accuracy, in its exact formulation.

Such calculations have been carried out at the Computing Center of the Academy of Sciences, USSR, on the electronic computing machine BESM-1.

The problem is posed of treating the calculation by a method that would be equally suitable for handling both plane and axisymmetric bodies of various shapes (smooth, with corners, combinations) with detached shock waves, for different values of the adiabatic index $\kappa (\kappa > 1)$ and of the free-stream Mach number $(1 < M_\infty \leqslant \infty)$. The plane problem was considered by the author [3], and calculation of flows at $M_\infty = 1$ was carried out by Chushkin [4]. The calculation scheme and also results of computations for certain simple shapes (ellipsoids, spheres and disks) are given below.

1. We consider flow with a detached shock wave past a body possessing axial symmetry. Let a supersonic stream of ideal gas flow with constant speed w_∞ past such a body at zero angle of attack. A shock wave forms ahead of the body, whose shape and location are initially unknown.

It is required to carry out the calculation of the mixed rotational gas flow in the minimal region of influence.

We introduce dimensionless variables, referring the speed w to the

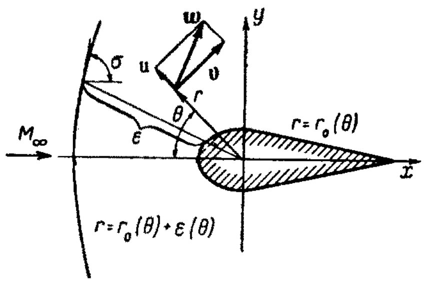

Fig. 1.

maximum speed w_{max}, the density ρ to the density ρ_∞ of the free stream (we will denote all quantities ahead of the shock wave by the index ∞), the pressure p to $\rho_\infty w_{max}{}^2$, and linear dimensions to the characteristic dimension of the body; then the system of gasdynamic equations will, in dimensionless form, be the following:

$$\operatorname{rot}\mathbf{w}\times\mathbf{w}+\frac{\nabla w^2}{2}+\frac{\nabla p}{\rho}=0,\qquad \nabla(\rho\mathbf{w})=0,\qquad \mathbf{w}\nabla\frac{p}{\rho^\varkappa}=0 \tag{1}$$

Henceforth the system of Equations (1) will be written for axisymmetric flow in spherical $(r,\ \theta)$ or Cartesian $(x = -r\cos\theta,\ y = r\sin\theta)$ coordinates (Fig. 1). We introduce the Bernoulli integral and the stream function ψ; then the full system of equations, for example in coordinates $(r,\ \theta)$, will have the form

$$\frac{\partial\,[r^2(p+\rho u^2)\sin\theta]}{\partial r}+\frac{\partial\,[r(\rho uv)\sin\theta]}{\partial\theta}=r(2p+\rho v^2)\sin\theta \tag{2}$$

$$\frac{\partial(r\tau u\sin\theta)}{\partial r}+\frac{\partial(r\tau v\sin\theta)}{\partial\theta}=0,\qquad \frac{d\psi}{d\theta}=r\rho\left(v\frac{dr}{d\theta}-ru\right)\sin\theta,\qquad \varphi=\varphi(\psi)$$

Here

$$p=\frac{\varkappa-1}{2\varkappa}(1-w^2),\qquad \rho=\tau\varphi^{-\frac{1}{\varkappa-1}},\qquad \tau=(1-w^2)^{\frac{1}{\varkappa-1}},\qquad \varphi=\frac{p}{\rho^\varkappa}$$

ϕ is the entropy function (entropy $s = c_v \ln\phi$), and u, v the components of the velocity w along r and θ. The unknown functions are u, v, ϕ, ψ. As boundary conditions we have the following.

On the body $r = r_0(\theta)$

$$u=\frac{v}{r}\frac{dr}{d\theta},\qquad \psi=0,\qquad \varphi=\varphi(0)=\text{const} \tag{3}$$

where

$$\varphi(0)=\frac{2}{\varkappa+1}\left(\frac{\varkappa-1}{\varkappa+1}\right)^{\varkappa}\frac{1}{w_\infty{}^{2\varkappa}}\left[w_\infty{}^2-\frac{(\varkappa-1)^2}{4\varkappa}(1-w_\infty{}^2)\right]$$

On the shock wave $r = r_0(\theta) + \epsilon(\theta)$

$$w_x = w_\infty\left[1 - \frac{2}{\varkappa+1}(\sin^2\sigma - m_\infty^2)\right], \quad w_y = (w_\infty - w_x)\operatorname{ctg}\sigma \qquad \left(m_\infty = \frac{1}{M_\infty}\right)$$

$$u = w_y \sin\theta - w_x\cos\theta, \qquad v = w_x\sin\theta + w_y\cos\theta$$

$$p = \frac{2}{\varkappa+1}\left[w_\infty^2\sin^2\sigma - (1 - w_\infty^2)\frac{(\varkappa-1)^2}{4\varkappa}\right] \tag{4}$$

$$\rho = \frac{\varkappa+1}{\varkappa-1}\frac{w_\infty^2\sin^2\sigma}{1 - w_\infty^2\cos^2\sigma}, \qquad \psi = \frac{(r_0+\varepsilon)^2}{2} w_\infty \sin^2\theta$$

Here ϵ is the distance along the arc θ = const from the body contour to the shock wave, σ is the angle of inclination of the wave to the free-stream direction (Fig. 1), and w_x and w_y are the components of the velocity w along x and y.

From (4) it is easy to obtain boundary conditions for ψ; furthermore from the relation $dy/dx = \tan\sigma$ on the shock wave we have

$$\frac{d\varepsilon}{d\theta} = -(r_0+\varepsilon)\cot(\sigma+\theta) - \frac{dr_0}{d\theta} \tag{5}$$

2. The method of Dorodnitsyn reduces the integration of a system of partial differential equations to the numerical solution of a certain approximating system of ordinary differential equations. The region of integration is here divided just as in the plane case [3]; N-1 intermediate lines are introduced between the body and the shock

$$r_i = r_0(\theta) + \xi_i\varepsilon(\theta), \qquad \xi_i = \frac{N-i+1}{N} \qquad (i = 2, 3, \ldots, N)$$

after which all the partial differential equations of the initial system are integrated along θ = const (or y = const) from the body contour to the boundary of each of the strips, and the remaining ordinary equations or finite relations of the system are written along the intermediate lines. Replacing the integrands by interpolation polynomials and integrating, we obtain an approximating system where the unknowns are the values of the dependent variables on the boundaries of the strips. We denote all quantities on the ith intermediate line by index i, on the shock wave (i = 1) by index 1, and on the body by index 0. Then the approximating system for (2) in the Nth approximation may be written schematically as

$$\frac{d\varepsilon}{d\theta} = -(r_0+\varepsilon)\cot(\sigma+\theta) - \frac{dr_0}{d\theta}$$

$$\frac{dr_0}{d\theta} = \frac{E_0}{D}, \qquad \frac{d\sigma}{d\theta} = \frac{E_\sigma}{D}, \qquad \frac{d\psi_i}{d\theta} = r_i\rho_i\left(v_i\frac{dr_i}{d\theta} - r_iu_i\right)\sin\theta \tag{6}$$

$$\frac{du_i}{d\theta} = U_i, \qquad \frac{dv_i}{d\theta} = \frac{E_i}{(\varkappa-1+2u_i^2)/(\varkappa+1) - w_i^2}$$

$$\varphi_i(\psi_i) = \varphi_1(\psi_i) \qquad (i = 2, 3, \ldots, N)$$

Here E_0, E_σ, U_i, E_i are definite holomorphic functions in the region of integration (the form of the functions depending on N), D is a known function $[D = 0$ in the neighborhood of $w_0^2 = (\kappa - 1)/(\kappa + 1)]$, whereby all boundary conditions on the body and on the shock wave are satisfied exactly in any approximation. In x, y coordinates the approximating system will have an analogous form.

The problem is thus reduced to finding the numerical solution of a boundary-value problem for the system (6) of ordinary differential equations, where part of the boundary conditions are given on the axis of symmetry:

$$v_0 = v_i = 0, \qquad \psi_i = 0, \qquad \sigma = \tfrac{1}{2}\pi, \qquad \varphi = \varphi_1(0) \qquad \text{for } \theta = 0$$

and the remainder on the singular line*, where we must have

$$E_0 = 0 \quad \text{for } D = 0, \qquad\qquad E_i = 0 \quad \text{for } w_i^2 = (\varkappa - 1 + 2u_i^2)/(\varkappa + 1)$$

Otherwise this singular line would be a limit line, and the solution would not have physical significance.

On the singular line the N equations of the system have N movable singular points of saddle type. The investigation is carried out analogously to the plane case (the Puisot diagram has just the same form). In the neighborhood of each singular point there exist two and only two solutions passing through the point, since both solutions are holomorphic. One of these solutions is "pasted together" with the solution up to the singular point obtained in the usual way (the number of pastings is equal to the degree of arbitrariness in the singular point), but since both solutions intersect only at the singular point, the pasting condition uniquely determines the integral curve passing through the given singular point. Some equations of the approximating system have singularities also on the axis of symmetry. However these are fixed singular points of regular type, from which it is possible to proceed with the use of power series in θ or y.

It is possible to show that in the axisymmetric case the formula for the angle of inclination of the line w = const to the free-stream direction at points on the shock wave has the following form:

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* When $D = 0$ and $E_0 = 0$, then also $E_\sigma = 0$ automatically.

$$\tan \Phi_1 = -\frac{[m(w_x - w_y \cot \sigma) + (w_\infty - w_x)(F_1 + F_2 \cot \sigma)]\, d\sigma / dy - n}{[(F_2 - m \cot \sigma)(w_x - w \cot \sigma) + w_y (F_3 - F_1)]\, d\sigma / dy + n \cot \sigma} \qquad (7)$$

Here

$$m = \frac{1}{\Delta} [(c^2 - w_y^2)(F_1 + F_2 \cot \sigma - F_3) + w_x (w_\infty - w_x)(F_1 + 2F_2 \cot \sigma)]$$

$$n = \frac{1}{\Delta y_1(x)} c^2 (w_\infty - w_x)(w_x - w_y \cot \sigma)$$

$$\Delta = c^2 (1 + \cot^2 \sigma) - (w_x - w_y \cot \sigma)^2$$

$$F_1 = \frac{1}{2\varkappa} (1 - w^2)^{\frac{\varkappa}{\varkappa - 1}} \varphi^{-\frac{1}{\varkappa - 1}} \frac{\sin 2\sigma}{w_\infty} \left\{ \frac{2w_\infty^2}{p(\varkappa + 1)} - \frac{2m_\infty^2 \varkappa (\varkappa + 1)}{p\,[2m_\infty^2 + (\varkappa - 1)\sin^2 \sigma]^2} \right\}$$

$$F_2 = -\frac{4w_\infty}{\varkappa + 1} \sin^2 \sigma, \qquad F_3 = \frac{2w_\infty}{\varkappa + 1} \left(\cos 2\sigma + \frac{m_\infty^2}{\sin^2 \sigma} \right) \tan \sigma, \qquad \left(m_\infty = \frac{1}{M_\infty} \right)$$

where $y = y_1(x)$ is the equation of the contour of the shock wave. Although in contrast to the plane case Φ_1 depends also on the curvature of the shock wave, and cannot be tabulated in advance, Formula (7) is nevertheless very useful, because it provides a possibility for estimating the accuracy of the solution.

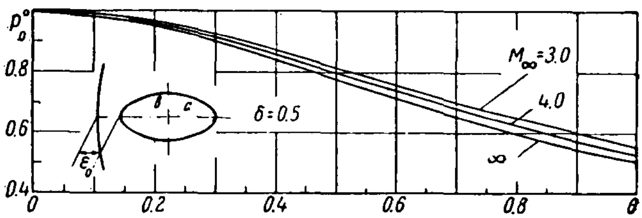

Fig. 2.

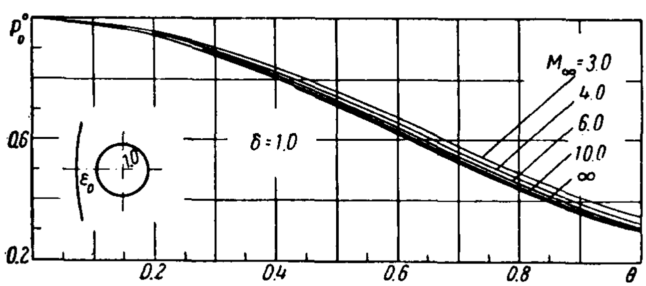

Fig. 3.

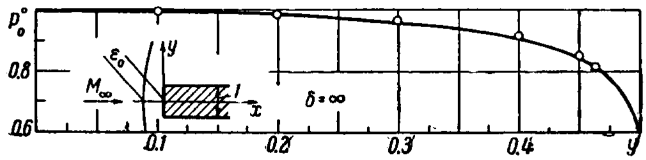

Fig. 4.

3. We give some results of the calculations. Solutions were carried out on the electronic computing machine BESM-1 for the flow past ellipsoids of revolution (with ratios of vertical to horizontal semi-axes $\delta = b/a = 0.5$ and $\delta = 1.5$), spheres ($\delta = 1.0$), and a flat-faced body ($\delta = \infty$) at various free-stream Mach numbers ($M_\infty = 3, 4, 6, 10, \infty$), $\kappa = 1.4$, and in various approximations ($N = 1, 2$).

The computing scheme for bodies with finite values of δ was constructed in r, θ coordinates (the origin was located at the center of curvature of the nose of the body, and all linear dimensions referred to the radius of curvature of the nose), whereas for the flat-faced body (disk) a Cartesian system of coordinates was used (linear dimensions being referred to the diameter of the disk).

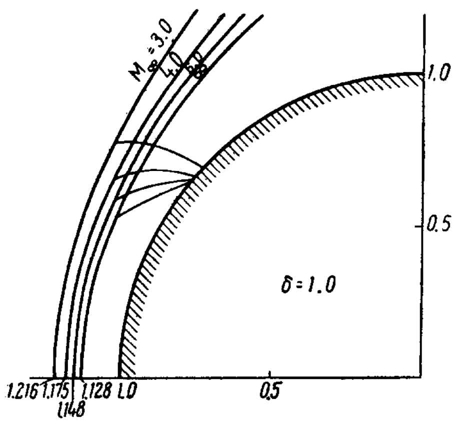

Fig. 5.

In Figs. 2-4 are given the distributions of pressure $p_0^0 = p_0(\theta)/p_0(0)$ over various bodies: in Fig. 2 for the ellipsoid with $\delta = 0.5$, in Fig. 3 for the sphere with $\delta = 1.0$, and in Fig. 4 for the disk with $\delta = \infty$, with $M_\infty = 3, 4, 10$ and ∞, where comparison is shown with experiments carried out by scientific colleague Shul'gin.

In Fig. 5 are shown the shock wave and sonic line for the sphere at

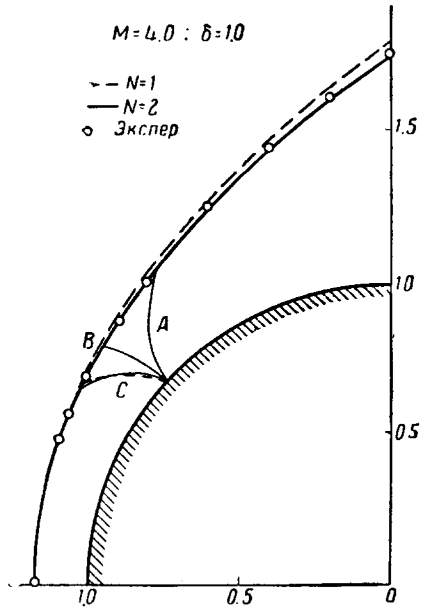

Fig. 6.

M_∞ = 3, 4, 6 and ∞ (with N = 1 and 2). The angles of intersection of the sonic line with the shock wave agree well with the values obtained from Equation (7).

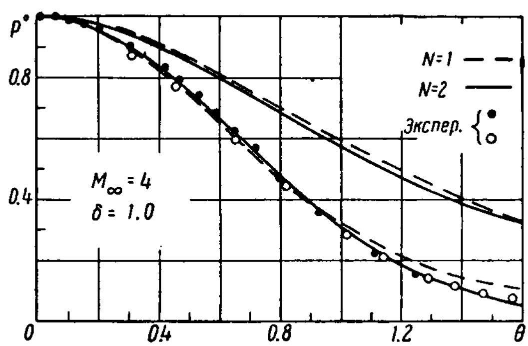

Fig. 7.

Figures 6 and 7 illustrate the convergence of the method with the degree of approximation. In Fig. 6 the curves A and B are characteristics of the first and second families, and curve C the sonic line. In Fig. 7 is given the pressure distribution on the shock wave (the two upper curves) and on the body (lower curves), and the results of experiments carried out by Shul'gin. In Fig. 8 it is shown how the distance from the body to the shock on the axis of symmetry varies for different δ.

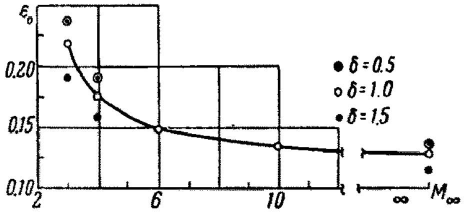

Fig. 8.

Numerical results are tabulated below for the solution for flow past a sphere at $M_\infty = 4.0$ and $\kappa = 1.40$ with $N = 2$. One table gives values of the velocity components u and v, the density ρ and pressure p at five points on the arcs θ = const: the distance from the body to the shock along a ray was divided into four equal parts

$$\xi = \frac{r-1}{\varepsilon(\theta)} = 0 \text{(body)},\ 0.25,\ 0.50,\ 0.75 \text{ and } 1.00 \text{ (shock)}$$

At $\xi = 1.0$, aside from these quantities, values are also given for ϵ, σ, the stream function ψ_1, the entropy function ϕ_1 (entropy $s_1 = c_v \ln\phi_1$) and the vorticity function $f_1 = d \ln \phi_1 / d\,\psi_1$.

The vorticity at the shock wave is determined by

$$\Omega_1 = \frac{(r_0 + \varepsilon)\sin\theta}{2\varkappa}(1 - w^2)^{\frac{\varkappa}{\varkappa - 1}} \varphi_1^{-\frac{1}{\varkappa - 1}} f_1, \qquad c^2 = \frac{\varkappa - 1}{2}(1 - w^2)$$

The stream function on a ray θ = const can be found by integration of the ordinary differential equation

$$\frac{d\psi}{dr} = r\rho v \sin\theta$$

Also shown in the tables are the coordinates of the sonic line and of the limiting characteristics for this same case.

The solutions for the body with $\delta = \infty$ were carried out by scientific colleague Shulishnin. All preliminary calculations and processing of results was performed by Mel'tsis, Bykov and Vasil'ev. I take this opportunity to express thanks to these colleagues.

θ	u	v	ρ	p	u	v	ρ	p
		$\xi = 0$				$\xi = 0.25$		
0.0000		0.0000	5.016	0.717	—0.0478	0.0000	4.990	0.711
0.0625		0.0353	5.001	0.714	—0.0474	0.0378	4.978	0.708
0.1250		0.0706	4.954	0.704	—0.0470	0.0755	4.939	0.700
0.1875		0.106	4.877	0.689	—0.0464	0.113	4.875	0.686
0.2500		0.141	4.770	0.668	—0.0456	0.150	4.788	0.667
0.3125		0.176	4.636	0.642	—0.0444	0.187	4.677	0.644
0.3750		0.211	4.477	0.611	—0.0430	0.223	4.544	0.616
0.4375		0.245	4.296	0.577	—0.0411	0.258	4.392	0.585
0.5000		0.279	4.094	0.539	—0.0389	0.293	4.222	0.550
0.5625		0.313	3.876	0.499	—0.0361	0.327	4.037	0.514
0.6250		0.346	3.645	0.458	—0.0328	0.360	3.841	0.477
0.6875	$u = 0$	0.379	3.407	0.417	—0.0287	0,391	3.636	0.440
0,7500		0.411	3.157	0.375	—0.0241	0.422	3.422	0.401
0.8125		0.443	2.908	0.334	—0.0185	0.452	3.207	0.364
0.8750		0.473	2.661	0.295	—0.0121	0.480	2.991	0.329
0.9375		0.503	2.419	0.258	—0.00467	0.507	2.778	0.295
1.0000		0.532	2.184	0.224	0.00380	0.532	2.570	0.263
1.0625		0.560	1.960	0.192	0.0134	0.556	2.369	0.234
1.1250		0.586	1.748	0.164	0.0241	0.578	2.176	0,207
1.1875		0.612	1.551	0.138	0.0359	0.599	1.992	0.182
1,2500		0.637	1.369	1.116	0.0489	0.618	1.819	0.160
1.3125		0.660	1.202	0.0970	0.0631	0.635	1.657	0.140
		$\xi = 0.5$				$\xi = 0.75$		
0.0000	—0.0956	0.0000	4.902	0.694	—0.143	0.0000	4.761	0.666
0.0625	—0.0949	0.0405	4.893	0.692	—0.142	0.0433	4.755	0.664
0.1250	—0.0941	0.0807	4.864	0.684	—0.141	0.0865	4.736	0.658
0.1875	—0.0927	0.121	4.816	0.672	—0.139	0.129	4.704	0.648
0.2500	—0.0903	0.160	4.748	0.655	—0.136	0.171	4.660	0.634
0.3125	—0.0883	0.199	4.664	0.635	—0.132	0.212	4.604	0.617
0.3750	—0.0850	0.236	4.562	0.611	—0.126	0.252	4.536	0.596
0.4375	—0.0811	0.274	4.445	0.583	—0.120	0.291	4.459	0.574
0.5000	—0.0763	0.309	4.315	0.554	—0.112	0.329	4.372	0.549
0.5625	—0.0705	0.344	4.172	0.523	—0.103	0.365	4.276	0.523
0.6250	—0.0637	0.377	4.019	0.490	—0.0929	0.399	4.172	0.496
0.6875	—0.0558	0.409	3.859	0.458	—0.0810	0.431	4.063	0.469
0.7500	—0.0466	0.439	3.692	0,425	—0.0676	0.462	3.949	0.441
0.8125	—0.0360	0.468	3.522	0.392	—0.0524	0.490	3.831	0.414
0.8750	—0.0239	0.494	3.351	0.362	—0.0355	0.516	3.710	0.388
0.9375	—0.0103	0.519	3.181	0.332	—0.0168	0.541	3.589	0.363
1.0000	0.00507	0.542	3.013	0.304	0.00380	0.563	3.467	0.338
1.0625	0.0221	0.564	2.848	0.278	0.0262	0.582	3.346	0.316
1.1250	0.0409	0.583	2,688	0.253	0.0506	0.600	3.226	0.294
1.1875	0.0615	0.600	2.534	0.230	0.0767	0.615	3.107	0.273
1.2500	0.0838	0.615	2.386	0.210	0.105	0.628	2.991	0.254
1.3125	0.108	0.628	2.244	0.190	0.134	0.638	2.877	0.236

θ	u	v	ρ	p	ε	σ	ψ_1	φ_1	f_1
					$\xi=1.0$				
0.0000	—0.191	0.0000	4.572	0.629	0.175	1.571	0.0000	0.0750	—0.717
0.0625	—0.190	0.0463	4,569	0.628	0.176	1.520	0.00235	0.0748	—0.734
0.1250	—0.188	0.0927	4.560	0.623	0.177	1.470	0.00940	0.0744	—0.738
0.1875	—0.185	0.139	4.546	0.615	0.179	1.419	0.0211	0.0738	—0.735
0.2500	—0.180	0.184	4.526	0.604	0.182	1.369	0.0373	0.0729	—0.728
0.3125	—0.174	0.228	4.501	0.590	0,186	1.320	0.0580	0.0718	—0.718
0.3750	—0.166	0.270	4,470	0.574	0.191	1.272	0.0831	0.0706	—0.707
0.4375	—0.157	0.312	4.434	0.556	0.198	1.225	0.112	0.0691	—0.693
0.5000	—0.147	0.351	4.392	0.536	0.205	1.179	0.146	0.0676	—0.677
0.5625	—0.134	0.389	4.346	0.516	0.214	1.134	0.183	0.0659	—0.658
0.6250	—0.120	0.425	4.294	0.494	0.224	1.090	0.224	0.0642	—0.639
0.6875	—0.104	0.459	4.237	0.471	0.236	1.048	0.269	0.0624	—0.614
0.7500	—0.0870	0.490	4.176	0.448	0.250	1.008	0.317	0.0606	—0.586
0.8125	—0.0678	0.520	4.110	0.426	0.266	0.969	0.369	0.0589	—0.559
0.8750	—0.0468	0.547	4.040	0.404	0.284	0.932	0.444	0.0571	—0.529
0.9375	—0.0242	0.571	3.967	0.382	0.305	0.896	0.483	0.0554	—0.499
1.0000	0.0000	0.593	3.891	0.360	0.328	0.862	0.545	0.0538	—0.467
1.0625	0.0258	0.613	3.811	0.340	0.355	0.830	0.611	0.0522	—0.434
1.1250	0.0530	0.630	3.728	0.320	0.385	0.798	0.681	0.0507	—0.401
1.1875	0.0816	0.644	3.644	0.301	0.418	0.769	0.755	0.0493	—0.367
1.2500	0.111	0.656	3.558	0.283	0.457	0.741	0.834	0.0479	—0.333
1.3125	0.142	0.665	3.470	0.266	0.500	0.714	0.918	0.0467	—0.299

Sonic line		**Characteristic of family I**		**Characteristic of family II**	
$-x=0.736$	$+y=0.677$	$-x=0.736$	$+y=0.677$	$-x=0.736$	$+y=0.677$
0.764	0.690	0.756	0.701	0.760	0.697
0.799	0.699	0.773	0.729	0.786	0.715
0.829	0.702	0.785	0.761	0.813	0.731
0.870	0.701	0.794	0.797	0.841	0.747
0.880	0.700	0.798	0.816	0.870	0.761
0.903	0.695	0.801	0.845	0.900	0.774
0.925	0.690	0.801	0.873	0.931	0.786
0.946	0.682	0.800	0.900	0.951	0.793
0.966	0.674	0.798	0.926		
0.986	0.665	0.791	0.977		
1.004	0.655	0.779	1.027		
1.022	0.645	0.773	1.051		
1.031	0.639				

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